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On the lack of semiconcavity of the subRiemannian distance in a class of Carnot groups *

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Abstract

We show by explicit estimates that the SubRiemannian distance in a Carnot group of step two is locally semiconcave away from the diagonal if and only if the group does not contain abnormal minimizing curves. Moreover, we prove that local semiconcavity fails to hold in the step-3 Engel group, even in the weaker "horizontal" sense.

1. Introduction

It is well known that subRiemannian spheres are rather irregular objects. Already in the simplest example—the Heisenberg group—the subRiemannian distance from the origin is only Lipschitz-continuous at points of the center of the group. Furthermore, it can be shown that the only subRiemannian manifolds where (small) spheres are smooth are the Riemannian ones (see [ABB16]).

The irregularity of the distance function is mainly governed by the presence of *abnormal geodesics* (see Section 2). Indeed, the function $d(x_0, \cdot)$ can not be smooth at any point x connected to x_0 by an abnormal length-minimizer (see [ABB16]). Furthermore, it has been shown in several papers by Agrachev, Bonnard, Chyba and Kupka [ABCK97], Trélat [Tré00] and Agrachev [Agr15] that, under the *corank 1* assumption, where in particular all abnormal extremals are *strictly abnormal*, at a point x along an abnormal length-minimizing curve γ leaving from x_0, the subRiemannian sphere centered at x_0 is tangent to γ in a suitable sense and ultimately the distance from x_0 can not be expected to be even Lipschitz at x.

On the other side, it is known that abnormal minimizers do not appear at all for a subclass of two-step Carnot groups (Métivier groups) and, by a result of Chitour, Jean and Trélat [CJT06], in the very large class furnished by *generic* subRiemannian structures of rank at least three.

In the papers [CR08,FR10], Cannarsa and Rifford, and Figalli and Rifford showed that in a bracket generating subRiemannian manifold where all length-minimizing paths are strictly normal, the subRiemannian distance from a fixed base point $x_0 \in M$ is locally semiconcave in $M \setminus \{x_0\}$. Since local semiconcavity implies local Lipschitz-continuity, this result can not be extended to the situation where corank 1 abnormal minimizers appear.

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However, there are subRiemannian manifolds and more specifically Carnot groups which do not belong to the class in [CR08, FR10], because they contain abnormal minimizing paths, but do not enjoy the corank 1 assumption of [Tré00] and [Agr15], because abnormal minimizing paths are normal too (we say that they are *normal-abnormal*). This class includes all non Métivier two-step Carnot groups and some step-three Carnot groups.

In this paper we show some negative results on the local semiconcavity of subRiemannian distances in the setting of non Métivier two-step groups and in the step-three Engel group. We also discuss a weaker property, namely the *horizontal semiconcavity* and we show that, in all two-step free groups, such property holds "pointwise" at all abnormal points, where the usual Euclidean notion fails to hold. We plan to come back to a detailed study of local horizontal semiconcavity for the distance in two-step Carnot groups in a subsequent work. On the other side, it turns out that in the three-step Engel group the horizontal semiconcavity fails to hold.

Besides its relevant role in the optimal transport problems studied in [FR10], local semiconcavity of the subRiemannian distance plays a role in the construction of suitable "barrier functions" in potential theory which are a fundamental tool in the study of second order nondivergence subelliptic PDEs with measurable coefficients (see [GT11], [Tra12], [Mon14]).

To state our result, we also introduce briefly some notation for two-step Carnot groups. Let (x,t) be coordinates in $\mathbb{R}^m \times \mathbb{R}^\ell$. Fix a family $A^1, \dots, A^\ell \in \mathbb{R}^{m\times m}$ of skew-symmetric matrices and define the composition law

$$(x,t)\cdot(\xi,\tau) = \Big(x+\xi, t+\tau+\frac{1}{2}\langle x, A\xi\rangle\Big) \tag{1.1}$$

where $\langle x, A\xi\rangle := (\langle x, A^1\xi\rangle), \dots, \langle x, A^\ell\xi\rangle) \in \mathbb{R}^\ell$ and $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^m$. We always assume the Hörmander condition $\operatorname{span}\{(A^1_{jk}, \dots, A^\ell_{jk}) : 1 \le j < k \le m\} = \mathbb{R}^\ell$ and we denote by d be the subRiemannian distance defined by the family of left-invariant vector fields $X_j = \partial_{x_j} + \frac{1}{2}\sum_{k=1}^m \sum_{\alpha=1}^\ell A^\alpha_{kj} x_k \partial_{t_\alpha}$, for $j = 1, \dots, m$. See Section 2.

Here is our statement on two-step Carnot groups, where we always denote by d the subRiemannian distance from the origin.

Theorem 1.1. *Let $(\mathbb{G},\cdot) = (\mathbb{R}^n,\cdot) = (\mathbb{R}^m_x \times \mathbb{R}^\ell_t, \cdot)$ be the two-step Carnot group equipped with the law* (1.1)*. Then, at any $(x,0) = \gamma(1)$, final point of an abnormal minimizer γ leaving from the origin, there are $C > 0$ and $\tau \in \mathbb{R}^\ell$ such that we have*

$$d(x,\beta\tau) - d(x,0) \ge C|\beta| \quad \text{for all } \beta \in [-1,1]. \tag{1.2}$$

Moreover, if $(\mathbb{G},\cdot) = (\mathbb{R}^n,\cdot)$ is free, then for any $(x,t) = \gamma(1)$, final point of an abnormal minimizer γ leaving from the origin, there are $C > 0$ and $(0,\tau) \in \mathbb{G}$ such that

$$d(x,t+\beta\tau) - d(x,t) \ge C|\beta| \quad \text{for all } \beta \in [-1,1]. \tag{1.3}$$

Remark that in two-step Carnot groups abnormal minimizers are always normal (see [AS04, Section 20.5] or [Rif14, Theorem 2.22]). Both estimates of this theorem ensure that the distance is not semiconcave (see the definition in (2.5)).

It is known that for step-two Carnot groups, $x \mapsto d(0,x)$ is Lipschitz for x belonging to compact sets which do not intersect the origin. Then, failure of semiconcavity can

be visualized as a presence of an outward Lipschitz cusp on a suitable “vertical section" of the sphere. Inner Lipschitz cusps do not conflict with semiconcavity (think of the Heisenberg group).

Our second result concerns the three-step Engel group $\mathbb{E} = \mathbb{R}^4$. In this setting any abnormal minimizer leaving from the origin is contained in a line ([Sus96, LS95]). The group law can be written in the form

$$x \cdot \xi = \Big(x_1 + \xi_1, x_2 + \xi_2, x_3 + \xi_3 + x_1\xi_2, x_4 + \xi_4 + \frac{x_1^2}{2}\xi_2 + x_1\xi_3\Big) \tag{1.4}$$

(see [BLU07, p. 285]) and the abnormal line containing the origin is $\{(0, x_2, 0, 0) \in \mathbb{R}^4 : x_2 \in \mathbb{R}\}$. We consider the control distance associated with the left-invariant vector fields

$$X_1 = \partial_1 \quad \text{and} \quad X_2 = \partial_2 + x_1\partial_3 + \frac{x_1^2}{2}\partial_4.$$

It follows from the results of [ABCK97] that the distance from the origin $d = d(0, \cdot)$ is not locally semiconcave at any point of such line. Here we prove a further result, showing that the distance is not even semiconcave in horizontal directions in any open set intersecting the abnormal line. Here is our result.

Theorem 1.2. *For all $x_2 \in \mathbb{R}$ there is $C > 0$ such that, if $|x_4|$ is small, then*

$$d(0, x_2, 0, x_4) - d(0, x_2, 0, 0) \geq C|x_4|. \tag{1.5}$$

Furthermore, we have the horizontal estimate

$$\limsup_{(y_1,y_2)\to 0} \frac{d\big(e^{y_1X_1+y_2X_2}(0, x_2, 0, 0)\big) - d\big((0, x_2, 0, 0)\big)}{y_1^2 + y_2^2} = +\infty. \tag{1.6}$$

The first inequality also follows from the estimate for Martinet vector fields proved in [ABCK97] (see Remark 4.1 below), but our proof is more elementary. To the best of our knowledge, estimate (1.6) is new.

Our arguments to estimate distances are not based on exact calculations with geodesics, which in some cases are rather difficult (see e.g. [AS11, AS15]). We use properties of minimizers to localize abnormal points and we estimate the distance from the origin of close points by elementary direct arguments.

The paper is structured as follows. Section 2 contains some general preliminaries. In Section 3 we discuss the step-two case and in Section 4 we discuss the Engel model.

2. General preliminaries

2.1. Control distances, endpoint maps and extremals

Let us start by recalling the vocabulary we will use in the following sections. For a complete discussion of the subject we refer to the monographs [AS04, ABB16, Rif14].

Given a family $X_1, \ldots, X_m$ of linearly independent smooth vector fields in $\mathbb{R}^n$, the subRiemannian distance associated with the family is defined as follows. An absolutely continuous path $\gamma \in W^{1,2}((0,1), \mathbb{R}^n)$ is said to be *horizontal* if there is a *control* $u \in$

$L^2((0,1),\mathbb{R}^m)$ such that we can write $\dot\gamma(t) = \sum_{j=1}^m u_j(t)X_j(\gamma(t))$ for a.e. $t \in (0,1)$. The subRiemannian length of a horizontal path γ is $\text{length}(\gamma) := \int_0^1 |u(t)|dt$. Given $x, y \in \mathbb{R}^n$, the subRiemannian distance between x and y is $d(x,y) = \inf\{\int_0^1 |u(t)|dt\}$, where the infimum is taken among all horizontal curves γ such that $\gamma(0) = x$ and $\gamma(1) = y$. If the Hörmander condition holds (i.e., the vector fields, together with their commutators of sufficiently large order span a space of dimension n at any point $x \in \mathbb{R}^n$) then for any pair of points $x, y \in \mathbb{R}^n$ there is a horizontal path connecting x, y and therefore $d(x,y)$ is finite. Furthermore, it turns out that for close points, the infimum is a minimum.

Given a fixed point $x_0 \in \mathbb{R}^n$, and given $u \in L^2((0,1),\mathbb{R}^m)$, we consider the a.e. solution γ_u of the nonautonomous Cauchy problem

$$\dot\gamma = \sum_j u_j(t)X_j(\gamma) \quad \text{with } \gamma_u(0) = x_0. \tag{2.1}$$

If $\gamma_u \in W^{1,2}((0,1),\mathbb{R}^n)$ is globally defined on $[0,1]$, we define the endpoint map $E(u) := \gamma_u(1)$. In Carnot groups, it turns out that the map $E : L^2 \to \mathbb{R}^n$ is globally defined and smooth. We say that γ has constant speed if $|u(s)|_{\mathbb{R}^m} = C$ for a.e. $s \in [0,1]$.

Let $x_0 \in \mathbb{R}^n$ be a fixed point and let $x \in \mathbb{R}^n$. Assume that there is a constant-speed path $\gamma : [0,1] \to \mathbb{R}^n$ which is a length minimizer between x_0 and x, i.e. $\text{length}(\gamma) = d(x_0, x)$. This implies that there is a nonzero vector $(\xi_0, \xi) \in \mathbb{R} \times \mathbb{R}^n$ such that

$$\xi_0 \langle u, v\rangle_{L^2} + \big\langle \xi, dE(u)v\big\rangle_{\mathbb{R}^n} = 0 \quad \forall\, v \in L^2 = L^2((0,1),\mathbb{R}^m), \tag{2.2}$$

where the linear map $dE(u) : L^2((0,1),\mathbb{R}^m) \to \mathbb{R}^n$ denotes the differential of E. If (2.2) holds, we say that u is an *extremal control*, or that the corresponding curve γ_u given by (2.1) is an *extremal curve*. Clearly, it suffices to consider the case $\xi_0 = 1$ and $\xi_0 = 0$. If (2.2) holds for some (ξ_0, ξ) with $\xi_0 = 1$, then we say that u is a *normal extremal control*, and γ_u is a *normal extremal curve*. If instead (2.2) holds for some (ξ_0, ξ) with $\xi_0 = 0$, then we say that u (resp. γ_u) is an *abnormal* extremal control (resp. curve). Equivalentely, abnormal controls are those controls $u \in L^2$ such that $dE(u) : L^2 \to \mathbb{R}^n$ is not open; they are sometimes called *singular controls* and the corresponding curves are called *singular curves*. The choice of (ξ_0, ξ) is not unique, and it may happen that a control is both normal and abnormal. In such case we say that u is normal-abnormal. If $\gamma = \gamma_u$ is an abnormal curve, the set of $(\lambda_0, \lambda) \in \mathbb{R} \times \mathbb{R}^n$ such that (2.2) holds is a subspace whose dimension is called the *corank* of γ (see [Tré00, Agr15]). Corank 1 extremals can not be normal-abnormal. Finally, a normal control/curve which is not abnormal is called *strictly normal* and an abnormal control/curve which is not normal is called *strictly abnormal*.

It is known that all abnormal length minimizing curves in two-step Carnot groups cannot be strictly abnormal (see [AS04]).

2.2. Two-step groups and Métivier condition

Let $\mathfrak{g} = V_1 \oplus V_2$ be a two-step nilpotent stratified Lie algebra (i.e. $[V_1, V_1] = V_2$ and $[\mathfrak{g}, V_2] = 0$). Let $\langle\cdot,\cdot\rangle_{V_1}$ be an inner product on V_1. Fix an orthonormal basis $X_1, \dots, X_m$ of V_1 and any basis $T_1, \dots, T_\ell$ of V_2. Then we have the commutation relations $[X_j, X_k] = \sum_{\alpha=1}^\ell A^\alpha_{jk} T_\alpha$ for suitable constants $A^\alpha_{jk} = -A^\alpha_{kj} \in \mathbb{R}$. Since $\text{Exp} : \mathfrak{g} \to \mathbb{G}$ is a global

diffeomorphism, we can identify the Lie group $\mathbb{G} = \operatorname{Exp}(\mathfrak{g})$ with $\mathbb{R}^m \times \mathbb{R}^\ell$ via exponential coordinates of the first kind

$$\mathbb{R}^m \times \mathbb{R}^\ell \ni (x_1, \dots, x_m, t_1, \dots, t_\ell) \simeq \operatorname{Exp}\Big(\sum_j x_j X_j + \sum_\alpha t_\alpha T_\alpha\Big) \in \mathbb{G} = \exp(\mathfrak{g}) \tag{2.3}$$

Finally, an application of the Baker–Campbell–Hausdorff–Dynkin formula (see [BLU07]) shows that the group law in $\mathbb{G}$ in the coordinates $(x,t) \in \mathbb{R}^m \times \mathbb{R}^\ell$ takes the form

$$(x,t) \cdot (y,s) = \Big(x + y, t + s + \frac{1}{2}\langle x, Ay\rangle\Big) \tag{2.4}$$

mentioned in (1.1). A subRiemannian frame of orthonormal horizontal left-invariant vector fields in given by $X_j = \partial_{x_j} + \frac{1}{2}\sum_{k=1}^m \sum_{\alpha=1}^\ell A_{kj}^\alpha x_k \partial_{t_\alpha}$, for $j = 1, \dots, m$. Moreover, $[X_j, X_k] = A_{jk} = \sum_\alpha A_{jk}^\alpha \partial_\alpha$. We assume the Hörmander condition $\operatorname{span}\{A_{jk} : 1 \leq j < k \leq m\} = \mathbb{R}^\ell$.

In a two-step group, given $\eta \in V_2^*$, define $J_\eta : V_1 \to V_1$ by the formula $\langle J_\eta X, X'\rangle = \eta([X, X'])$. We say that the group satisfies the Métivier condition [Mét80] if the linear map J_η is an isomorphism for all $\eta \in V_2^* \setminus \{0\}$. The Métivier class includes the class of the groups of Heisenberg type (with strict inclusion, see [MS04, Section 7] or [BLU07]). An equivalent way to state the Métivier condition is by requiring that the map $\mathbb{R}^m \ni y \mapsto \langle Aw, y\rangle \in \mathbb{R}^\ell$ is onto for all $w \in \mathbb{R}^m \setminus \{0\}$. Another equivalent assumption is that the square matrix $\sigma A := \sum_{\alpha=1}^\ell \sigma_\alpha A^\alpha \in \mathbb{R}^{m \times m}$ is nonsingular for all $\sigma = (\sigma_1, \dots, \sigma_\ell) \neq 0 \in \mathbb{R}^\ell$.

2.3. Semiconcavity

Following [CS04, Definition 1.1.1] and [FR10], we say that a continuous function $f : \Omega \to \mathbb{R}$ is *semiconcave* on the open set $\Omega \subset \mathbb{R}^n$ if there is $C > 0$ such that

$$f(x+h) + f(x-h) - 2f(x) \leq 2C|h|^2, \tag{2.5}$$

for all $x, h \in \mathbb{R}^n$ such that the segment $[x-h, x+h]$ is contained in Ω. Equivalently, there is $C > 0$ so that

$$\lambda f(y) + (1-\lambda) f(x) - f(\lambda y + (1-\lambda)x) \leq C\lambda(1-\lambda)|x-y|^2$$

for all x, y such that $[x,y] \subset \Omega$ and $\lambda \in [0,1]$. Roughly speaking, second order derivatives of a semiconcave function can be $-\infty$, but they must be bounded from above by some positive constant $C < \infty$. See [CS04, Chapter 2].

The following theorem has been shown by Cannarsa and Rifford [CR08], and Figalli and Rifford [FR10]:

Theorem 2.1. *Let M be a subRiemannian manifold with subRiemannian distance d. Let $x_0 \in M$ and assume that for all $y \in M$ every length minimizing path connecting x_0 and y is nonsingular. Then, the distance function $y \mapsto d(x_0, y)$ is locally semiconcave on $M \setminus \{x_0\}$.*

3. Step-two groups

3.1. Some (mostly known) facts on step-two groups

3.1.1 Endpoint map and extremal paths

Let $\mathbb{R}^m \times \mathbb{R}^\ell$ be equipped with the group law (2.4). Denote by $e = (0,0)$ the identity element of the group and by $d(x,t)$ the distance from the origin of $(x,t) \in \mathbb{R}^m \times \mathbb{R}^\ell$. The ODE for the curve $\gamma = (x,t)$ associated with a control $u \in L^2((0,1),\mathbb{R}^m)$ is

$$\dot{x}(s) = u(s) \qquad \dot{t}(s) = \frac{1}{2}\langle x(s), Au(s)\rangle, \qquad \text{with } (x(0),t(0)) = (0,0) \tag{3.1}$$

where $\langle x, Au\rangle = (\langle x, A^1 u\rangle, \dots, \langle x, A^\ell u\rangle)$. Given $u \in L^2(0,1)$, the endpoint map $E(u) = \gamma(1) = (x(1),t(1))$ has the form

$$E(u) = \Big(\int_0^1 u(s)ds,\ \frac{1}{2}\int_0^1 \Big\langle \int_0^s u, Au(s)\Big\rangle ds\Big).$$

As calculated in [AGL15], its differential $dE(u) : L^2 \to \mathbb{R}^m \times \mathbb{R}^\ell$ has the following form

$$\begin{aligned} dE(u)v &= \Big(\int_0^1 v, \frac{1}{2}\int_0^1 \Big\{\Big\langle \int_0^s u, Av(s)\Big\rangle + \Big\langle \int_0^s v, Au(s)\Big\rangle\Big\} ds\Big) \\ &= \Big(\int_0^1 v,\ \int_0^1 \Big\langle A\Big(\frac{x}{2} - \int_0^s u\Big), v(s)\Big\rangle ds\Big). \end{aligned} \tag{3.2}$$

We integrated by parts and we let $\int_0^1 u = x$.

Next, we recapitulate the discussion in [AGL15]. Let $u \in L^2(0,1)$ be a minimizing control for the problem $\min\{\|u\|^2_{L^2(0,1)} : E(u) = (x,t)\}$. Since minimizing controls in step-two Canot groups are always normal (this follows from the second order analysis of the Goh condition, see [AS04, Section 20.5] or [Rif14, Theorem 2.22]), there is a nontrivial (co)vector $(\xi,\tau) \in \mathbb{R}^m \times \mathbb{R}^\ell$ such that

$$\begin{aligned} 0 &= \langle u, v\rangle_{L^2} - \langle(\xi,\tau), dE(u)v\rangle \\ &= \int_0^1 \langle u(s), v(s)\rangle ds - \int_0^1 \langle \xi, v(s)\rangle ds - \int_0^1 \Big\langle \tau A\Big(\frac{x}{2} - \int_0^s u\Big), v(s)\Big\rangle \qquad \text{for all } v \in L^2(0,1). \end{aligned}$$

Here $\tau A := \sum_{\alpha=1}^\ell \tau_\alpha A^\alpha = -(\tau A)^T \in \mathbb{R}^{m\times m}$. Since $v \in L^2$ is arbitrary, we get

$$u(s) = \xi + \tau A\frac{x}{2} - \tau A\int_0^s u(\rho)d\rho \quad \text{for all } s \in [0,1]. \tag{3.3}$$

Therefore, $\dot{u}(s) = -\tau A u(s)$ and then, according to [AGL15, Proposition 5],

$$u(s) = e^{-\tau A s}u, \tag{3.4}$$

for a suitable $u \in \mathbb{R}^m$. It is easy to recognize that, since A is skew symmetric, then $e^{-\tau A s} \in O(m)$ is an orthogonal $m \times m$ matrix. Therefore, the path γ has constant speed and length$(\gamma) = |u|$.

Let $u \in L^2(0,1)$ be an abnormal extremal. Then by definition there is $(\eta, \sigma) \in \mathbb{R}^m \times \mathbb{R}^\ell \setminus \{(0,0)\}$ such that

$$0 = \langle (\eta, \sigma), dE(u)v \rangle = \int_0^1 \Big\langle \eta + \sigma A\Big(\frac{x}{2} - \int_0^s u\Big), v(s) ds \Big\rangle \quad \text{for all } v \in L^2(0,1).$$

Since v is arbitrary, one gets

$$\eta + \sigma A \frac{x}{2} - \sigma A \int_0^s u = 0 \quad \text{for all } s \in [0,1], \tag{3.5}$$

with the usual convention $\sigma A := \sum_{\alpha=1}^{\ell} \sigma_\alpha A^\alpha$. Note that it must be $\sigma \neq 0$. Otherwise (η, σ) becomes trivial. Differentiating we obtain, according with [Kis03, Lemma 2.4] and [Hsu92] the condition

$$\sigma A u(s) = 0 \quad \text{for almost all } s \tag{3.6}$$

(which implies $\eta = 0$). Since $\ker(\sigma A)$ is a subspace of dimension at most $m-2$, the structure of the ODE (3.1) implies that, letting $\mathrm{Abn}(e) = \{\gamma(1) : \gamma \text{ is abnormal and } \gamma(0) = e\}$ we have

$$\mathrm{Abn}(e) \subseteq \bigcup \Big\{ \mathbb{G}_W : W \text{ subspace of } \mathbb{R}^m, \quad \dim W \le m-2 \Big\} \tag{3.7}$$

where $\mathbb{G}_W$ is the subgroup

$$\mathbb{G}_W := \mathrm{span}\,\big\{ \big(w, \langle w', A w'' \rangle\big) : w, w', w'' \in W \big\}, \tag{3.8}$$

which is a Carnot group of step $r \in \{1,2\}$. To check this claim, note that (3.6) ensures that there is a subspace $W \subset \mathbb{R}^m$ of dimension at most $m-2$ such that $u(s) \in W$ a.e. in $s \in [0,1]$. Then $x(s) = \int_0^s u \in W$ and $t(s) = \frac{1}{2}\int_0^s \langle x(\rho), A u(\rho) \rangle d\rho \in \mathrm{span}\{\langle w', A w'' \rangle : w', w'' \in W\}$. The inclusion (3.7) can be strict, but it is an equality for free groups (see [DMO+15] and Remark 3.2 below).

Furthermore, (3.6) implies that a control of the form $u(s) = e^{-\tau A s} u$ is abnormal if and only if there is $\sigma \in \mathbb{R}^\ell \setminus \{0\}$ such that

$$\sigma A (\tau A)^m u = 0 \qquad \text{for all } m \in \mathbb{N} \cup \{0\}. \tag{3.9}$$

It may happen that $\sigma \in \mathrm{span}\{\tau\}$ and in such case, comparing (3.3) and (3.5), we see that $u(s) = e^{-\tau A s} u = u \in \ker \tau A$ is a constant control.

3.1.2 Bivectors and skew-symmetric matrices

If we denote by $e_1, \dots, e_m$ the canonical basis of $\mathbb{R}^m$, we define $\wedge^2 \mathbb{R}^m := \mathrm{span}\{e_j \wedge e_k : 1 \le j < k \le m\}$. Given two vectors $x, y \in \mathbb{R}^m$, the elementary bivector $z = x \wedge y \in \wedge^2 \mathbb{R}^m$ can be expanded as

$$x \wedge y = \sum_j (x_j e_j) \wedge \sum_k (y_k e_k) = \sum_{1 \le j < k \le m} (x_j y_k - x_k y_j) e_j \wedge e_k =: \sum_{1 \le j < k \le m} (x \wedge y)_{jk} e_j \wedge e_k.$$

On $\wedge^2 \mathbb{R}^m$ we define the standard inner product on elementary bivectors

$$\langle x \wedge y, \xi \wedge \eta \rangle = \langle x, \xi \rangle \langle y, \eta \rangle - \langle x, \eta \rangle \langle y, \xi \rangle \quad \text{for all } x, y, \xi, \eta \in \mathbb{R}^m.$$

This is equivalent to the requirement that the family $e_j \wedge e_k$, with $1 \leq j < k \leq m$, is orthonormal in $\wedge^2\mathbb{R}^m$. The inner product $\langle z, \zeta\rangle$ can be extended by linearity to general bivectors $z = \sum_{a=1}^{n} x_a \wedge y_a$ and $\zeta = \sum_{\alpha=1}^{\nu} \xi_\alpha \wedge \eta_\alpha$, for any $x_a, y_a, \xi_\alpha, \eta_\alpha \in \mathbb{R}^m$. Note that if $\mathbb{R}^m = V \oplus W$ decomposes as a sum with $V \perp W$ and we choose orthonormal bases $v_1, \dots, v_p$ of V and $w_1, \dots w_q$ of W, it turns out that the family $\{v_j \wedge v_k, v_j \wedge w_\alpha, w_\alpha \wedge w_\beta : 1 \leq j < k \leq p, \quad 1 \leq \alpha < \beta \leq q\}$ is an orthonormal basis of $\wedge^2\mathbb{R}^m$ and ultimately the three terms in the decomposition

$$\wedge^2\mathbb{R}^m = \wedge^2 V \oplus (V \wedge W) \oplus \wedge^2 W \tag{3.10}$$

are pairwise orthogonal. Here and hereafter we are keeping the short notation $V \wedge W := \operatorname{span}\{v \wedge w : v \in V, w \in W\}$.

Let $M = -M^T \in \mathbb{R}^{m\times m}$ be a skew-symmetric matrix of rank $2p \leq m$. By spectral theory, there are p two-dimensional pairwise orthogonal subspaces $V_1, \dots V_p$, p positive numbers $\lambda_1, \dots, \lambda_p > 0$ and a corresponding orthonormal basis $v_h, v_h^\perp$ of each V_h such that

$$Mv_h = \lambda_h v_h^\perp \quad \text{and} \quad Mv_h^\perp = -\lambda_h v_h \quad \text{for all } h = 1, \dots, p.$$

In other words, we can write $Mx = \sum_{h=1}^{p} \lambda_h \big(\langle x, v_h\rangle v_h^\perp - \langle x, v_h^\perp\rangle v_h\big)$. Observe that $\operatorname{Im} M = \oplus_{h=1}^{p} V_h$ and $\ker M = (\oplus_h V_h)^\perp$. It may happen that $\lambda_i = \lambda_j$ for some $i \neq j$. The generic element of M is $M_{jk} = \Big(\sum_{h=1}^{p} \lambda_h v_h^\perp \wedge v_h\Big)_{jk}$. The rank of the bivector $\sum_{h=1}^{p} \lambda_h v_h^\perp \wedge v_h \in \wedge^2\mathbb{R}^m$ is by definition p. Moreover, the space $\operatorname{span}\{v_h, v_h^\perp : 1 \leq h \leq p\}$ is called the *support* of the bivector.

A short computation shows that the exponential of M applied to $x \in \mathbb{R}^m$ is

$$\begin{aligned} e^M x = \sum_{h=1}^{p} \cos(\lambda_h)\big(\langle x, v_h\rangle v_h + \langle x, v_h^\perp\rangle v_h^\perp\big) + \sum_{h=1}^{p} \sin(\lambda_h)\big(\langle x, v_h\rangle v_h^\perp - \langle x, v_h^\perp\rangle v_h\big) \\ + \Big\{x - \sum_{h=1}^{p}\big(\langle x, v_h\rangle v_h + \langle x, v_h^\perp\rangle v_h^\perp\big)\Big\}. \end{aligned} \tag{3.11}$$

3.1.3 Extremal curves in free groups

Let $\mathbb{F}_m \equiv \mathbb{F}_{m,2} := \mathbb{R}^m \times \wedge^2\mathbb{R}^m$ with the group law

$$(x,t)\cdot(\xi,\tau) = \Big(x + \xi, t + \tau + \frac{1}{2} x \wedge \xi\Big). \tag{3.12}$$

Here for convenience of notation we used $\wedge^2\mathbb{R}^m$ instead of $\mathbb{R}^\ell$ and we made the choice of matrices $A^{jk} \in \mathbb{R}^{m\times m}$ defined as follows: $A^{jk}x = x_k e_j - x_j e_k$. Then, for any $x, \xi \in \mathbb{R}^m$ we indicate with $\langle x, A\xi\rangle \in \wedge^2\mathbb{R}^m$ the bivector

$$\langle x, A\xi\rangle = x \wedge \xi = \sum_{1\leq j<k\leq m} (x \wedge \xi)_{jk} e_j \wedge e_k = \sum_{1\leq j<k\leq m} (x_j\xi_k - x_k\xi_j) e_j \wedge e_k. \tag{3.13}$$

Let $u(s) = e^{-\tau A s}u$ be a normal extremal control. Since $-\tau A$ is a skew-symmetric matrix, there are $p \leq \frac{n}{2}$, strictly positive numbers $\lambda_1, \dots, \lambda_p > 0$ and corresponding pairwise

orthogonal vectors $a_1, a_1^\perp, \dots, a_p, a_p^\perp, z$ such that

$$u(s) = \sum_{k=1}^{p} \big(\cos(\lambda_k s) a_k + \sin(\lambda_k s) a_k^\perp\big) + z, \tag{3.14}$$

where $|a_k| = |a_k^\perp| > 0$ for all $k = 1, \dots, p$ and $z \perp \operatorname{span}\{a_k, a_k^\perp : 1 \le k \le p\}$. Here it may be $z = 0$. The free-group assumption ensures that the matrix $-\tau A$ can be any skew-symmetric matrix and thus any control u of the form (3.14) is a normal extremal control.

Moreover, we may assume without loss of generality that in (3.14) the following "non-degeneration condition" holds

$$0 < \lambda_j \neq \lambda_k \quad \text{for all} \quad j \neq k. \tag{3.15}$$

Otherwise, if $\lambda_j = \lambda_k$ for some $j \neq k$, then we can write

$$\cos(\lambda_j s) a_j + \sin(\lambda_j s) a_j^\perp + \cos(\lambda_j s) a_k + \sin(\lambda_j s) a_k^\perp = \cos(\lambda_j s)(a_j + a_k) + \sin(\lambda_j s)(a_j^\perp + a_k^\perp).$$

Observe that if we add to condition (3.15) the requirement $\lambda_j < \lambda_k$ if $j < k$, then all the data $p, \lambda_k, a_k, a_k^\perp, z$ are uniquely determined by $u(s)$. Finally, the length of the curve γ_u corresponding to the control (3.14) is $\operatorname{length}(\gamma_u)^2 = |z|^2 + \sum_{k=1}^{p} |a_k|^2$. The curve corresponding to the extremal control (3.14) lives in the subgroup $W \times \wedge^2 W$, where

$$W := \operatorname{span}\{a_1, a_1^\perp, \dots, a_p, a_p^\perp, z\}. \tag{3.16}$$

The discussion below shows that γ_u is nonsingular in the subgroup $\mathbb{G}_W := W \times \wedge^2 W$. In general the inclusion $\mathbb{G}_W \subset dE(u)L^2$ is strict.

In order to characterize singular extremals, we will use the following linear algebra lemma.

Lemma 3.1. *Let $v_1, \dots, v_p \in \mathbb{R}^m$ and let $0 < \lambda_1 < \lambda_2 < \dots < \lambda_p$ be positive numbers. Then,*

$$\begin{aligned} &\operatorname{span}\{\lambda_1^{2k-1} v_1 + \lambda_2^{2k-1} v_2 + \dots + \lambda_p^{2k-1} v_p : 1 \le k \le p\} \\ &\quad = \operatorname{span}\{\lambda_1^{2k-1} v_1 + \lambda_2^{2k-1} v_2 + \dots + \lambda_p^{2k-1} v_p : k \in \mathbb{N}\} \\ &\quad = \operatorname{span}\{v_1, v_2, \dots, v_p\}. \end{aligned}$$

An analogous statement holds changing the powers $2k-1$ with $2k$.

Proof. In both equalities $\subseteq$ is trivial. To accomplish the proof, it suffices to show that the set in the first line contains $\operatorname{span}\{v_1, v_2, \dots, v_p\}$. To see this fact observe that

$$\begin{aligned} &[\lambda_1 v_1 + \dots + \lambda_p v_p \,|\lambda_1^3 v_1 + \dots + \lambda_p^3 v_p \,| \cdots |\lambda_1^{2p-1} v_1 + \dots + \lambda_p^{2p-1} v_p] \\ &\qquad = [v_1 | \cdots | v_p] \begin{bmatrix} \lambda_1 & \lambda_1^3 & \cdots \lambda_1^{2p-1} \\ \lambda_2 & \lambda_2^3 & \cdots \lambda_2^{2p-1} \\ \cdots & \cdots & \cdots \cdots \\ \lambda_p & \lambda_p^3 & \cdots \lambda_p^{2p-1} \end{bmatrix} \end{aligned}$$

The thesis follows because the Vandermonde matrix is nonsingular. $\square$

Next we recall the characterization of singular extremal controls (see also [DMO+15]). Let u be a normal extremal control of the form (3.14) satisfying the nondegeneration condition (3.15). Then, u is singular if and only if there is a nontrivial skew-symmetric matrix $\sigma A \in \mathbb{R}^{m\times m}$ such that $\sigma A u(s) = 0$ for all s. By properties of the kernel of skew-symmetric matrices this is equivalent to say that there is a $(m-2)$-dimensional subspace $W \subseteq \mathbb{R}^m$ such that $u(s) \in W$ for all s. Equivalently, $\dim\operatorname{span}\{u^{(k)}(0) : k \in \mathbb{N}\cup\{0\}\} \le m-2$, which means

$$\dim\operatorname{span}\Big\{z + \sum_{k=1}^{p} a_k, \sum_{k=1}^{p} \lambda_k a_k^{\perp}, \sum_{k=1}^{p} \lambda_k^2 a_k, \sum_{k=1}^{p} \lambda_k^3 a_k^{\perp}, \dots\Big\} \le m-2. \tag{3.17}$$

Since we assume (3.15), using the lemma above, it is easy to recognize that this is equivalent to the requirement

$$\dim\operatorname{span}\{a_1, a_1^{\perp}, \dots, a_p, a_p^{\perp}, z\} \le m-2 \tag{3.18}$$

Remark 3.2. *Formula* (3.18) *is related with the parametrization of the abnormal set provided in formula (3.9) in [DMO+15]. Indeed it implies that*

$$\operatorname{Abn}^{\rm nor}(e) = \operatorname{Abn}(e) = \bigcup\Big\{W \times \wedge^2 W : W \subset \mathbb{R}^m \quad \dim W = m-2\Big\} \tag{3.19}$$

where $\operatorname{Abn}^{\rm nor}(e)$ *indicates the endpoints of normal-abnormal curves leaving from the origin. The first* $\subseteq$ *inclusion is obvious and the second follows from* (3.7). *The fact that* $\operatorname{Abn}^{\rm nor}(e)$ *contains the union on the right-hand side can be seen as follows. Let* $W \subseteq \mathbb{R}^m$ *be a subspace of dimension* $\dim W = m-2$. *Then* $W \times \wedge^2 W$ *is isomorphic to the free two-step group with* $m-2$ *generators. Therefore, for each point* $(w,\xi) \in W \times \wedge^2 W$ *there is a control of the form* (3.14) *with* $a_1, a_1^{\perp}, \dots, a_p, a_p^{\perp}, z \in W$ *and such that the curve* γ *arising from such control connects the origin with* (w,ξ).

3.1.4 Extremals in general two-step groups

If $(\mathbb{R}^m \times \mathbb{R}^\ell, \cdot)$ is a two-step Carnot group with law (1.1), normal extremal curves can be described similarly to the free case, but there are some differences. Indeed, given an extremal control $u(s) = e^{-\tau A s}u$, while in the free case $-\tau A$ was the most general skew-symmetric matrix, here, as observed by [AGL15], the matrix $-\tau A$ should belong to the subspace of $\mathfrak{so}(m)$, generated by $A_1, \dots, A_\ell$. Anyway, applying spectral theory to the matrix $-\tau A$, we see that $u(s)$ can be written in the form

$$u(s) = \sum_{k=1}^{p} \cos(\lambda_k s) a_k + \sin(\lambda_k s) a_k^{\perp} + z, \tag{3.20}$$

where, as in the free case we assume without loss of generality the nondegeneration condition

$$0 < \lambda_j \ne \lambda_k \quad \text{for all} \quad j \ne k. \tag{3.21}$$

Again, making the further requirement $\lambda_j < \lambda_k$ if $j < k$, then all the data $p, \lambda_k, a_k, a_k^{\perp}, z$ are uniquely determined by $u(s)$. If we let, as in the free case $W := \operatorname{span}\{a_1, a_1^{\perp}, \dots, a_p, a_p^{\perp}, z\}$,

then by (3.1), it turns out that the curve corresponding to the extremal control (3.20) lives in the subgroup $\mathbb{G}_W$ introduced in (3.8).

The description of singular extremals is less precise than in the free case. However, by (3.6) and Lemma 3.1, we can say that a control of the form (3.20) under the nondegeneration condition is singular if and only if there is $\sigma \in \mathbb{R}^\ell$ such that the associated subspace W satisfies $W \subset \ker(\sigma A)$. Equivalently, there is $\sigma \neq 0$ such that $\sigma \perp \langle Aw, y\rangle$ in $\mathbb{R}^\ell$ for all $w \in W$ and $y \in \mathbb{R}^m$. This ensures that $W \subset \mathbb{R}^m$ has dimension at most $m-2$. Furthermore, under (3.21), it turns out that γ_u is nonsingular in the subgroup $\mathbb{G}_W$ defined in (3.8) (if it would be singular, then $\{u(s) : s \in [0,1]\}$ would be contained in a subspace of dimension at most $\dim W - 2$).

Remark 3.3. *The objects of the discussion above have a strict relation with the abnormal varieties Z^λ studied in [LDLMV13] and [DMO+15, Section 3.1]. Indeed, fixed the basis $X_1, \dots, X_m, T_1, \dots T_\ell$ of $\mathfrak{g} = V_1 \oplus V_2$ as in Section 2.2 and the dual basis $\eta_1, \dots, \eta_m, \theta_1, \dots, \theta_\ell$ of $\mathfrak{g}^* = V_1^* \oplus V_2^*$, then, choosing the covector $\lambda = \sum_\alpha \sigma_\alpha \theta_\alpha \in V_2^*$, a computation shows that, in the exponential coordinates* (2.3)

$$Z^\lambda = \{(x,t) \in \mathbb{R}^m \times \mathbb{R}^\ell : \sigma A x = 0\} = \ker(\sigma A) \times \mathbb{R}^\ell.$$

where $\sigma A = \sum_\alpha \sigma_\alpha A^\alpha$ as usual. Thus, $\mathfrak{z}^\lambda \cap V_1 = \{\sum_j x_j X_j : x \in \ker \sigma A\} \subset V_1$ and

$$H^\lambda = \operatorname{span}\{(x, \langle \xi, A\eta\rangle) : x, \xi, \eta \in \ker(\sigma A)\}$$

is the subgroup appearing in [DMO+15, Eq. (3.1)].

Next we calculate the image of the differential of the endpoint map at extremal controls in terms of the associated subspace W.

Proposition 3.4. *Let $u \in L^2(0,1)$ be a normal extremal control of the form* (3.20) *satisfying the nondegeneration condition* (3.21). *Then, if $W = \operatorname{span}\{a_1, a_1^\perp, \dots, a_p, a_p^\perp, z\}$, we have*

$$\operatorname{Im} dE(u) = \operatorname{span}\{(\xi, \langle Aw, \eta\rangle) : w \in W\ \xi, \eta \in \mathbb{R}^m\}.$$

Proof. Formula (3.2) immediately implies $\subseteq$.

To see $\supseteq$, we test formula (3.2) against sequences of smooth functions approximating the δ function and its derivatives of order $\ell \geq 1$. Precisely, let $\varphi \in C_c^\infty(]-1,1[)$ be a nonnegative averaging kernel with $\int_{-1}^1 \varphi(s)ds = 1$. Then define the family $(\varphi_n)_{n\geq 2}$, by $\varphi_n(s) := n\varphi(ns-1)$. It turns out that $\varphi_n \in C_c^\infty(]0,1[)$ and φ_n is an approximation of the Dirac mass at $s=0$ as $n \to \infty$. Moreover, for $\ell = 0,1,2,\dots$, $\xi \in \mathbb{R}^m$ and $n \in \mathbb{N}$, the family $(\varphi_n^\ell)_{n\geq 2}$, $\varphi_n^\ell(s) := (\frac{d}{ds})^\ell \varphi_n(s)$ approximates the ℓ-th derivative of the Dirac mass, as $n \to \infty$.

Let us take $\xi \in \mathbb{R}^m$ and define $v_n^\ell(s) = \varphi_n^\ell(s)\xi$. Testing (3.2) with $(v_n^0)_{n\in\mathbb{N}}$ and passing to the limit as $n \to \infty$ we find

$$\operatorname{Im} dE(u) \supseteq \Big\{\big(\xi, \Big\langle A\frac{x}{2}, \xi\Big\rangle\big) : \xi \in \mathbb{R}^m\Big\}.$$

If instead $\ell \geq 1$, calculating $dE(u)v_n^\ell$ and letting $n \to \infty$, we find

$$\operatorname{Im} dE(u) \supseteq \Big\{\big(0, -\langle Ax^{(\ell)}(0), \xi\rangle\big) : \xi \in \mathbb{R}^m\Big\} \quad \text{for all } \ell = 1,2,\dots.$$

The proof is easily concluded because $\operatorname{span}\{x^{(\ell)}(0) : \ell \geq 1\} = W$. [1] ☐

Remark 3.5. *In the nonfree case, it is not true that* $\operatorname{Abn}(e)$ *can be parametrized as* $\bigcup\{\mathbb{G}_W : \dim(W) \leq m-2\}$*, where* $\mathbb{G}_W$ *is the subgroup in* (3.8)*. A counterexample is given by a direct product* $\mathbb{H}_{x_1,x_2,t} \times \mathbb{R}_{x_3}$ *of the Heisenberg group with the Euclidean line. Here, for* $x, \xi \in \mathbb{R}^3$ *we define* $\langle x, A\xi\rangle = x_1\xi_2 - x_2\xi_1$ *In this case, for any* $w = (w_1, w_2, 0) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$*, a curve of the form* $\gamma(s) = (sw_1, sw_2, 0, 0)$ *is an extremal and is contained in the subgroup* $\mathbb{G}_W$ *where* $W = \operatorname{span}\{w\}$ *is one-dimensional. However,* γ *is nonsingular in the product.*

Using Proposition 3.4 it is easy to see that abnormal minimizing curves appear if and only if the Métivier condition fails. This statement is implicitly contained in [DMO+15, Eq. 3.2)].

Proposition 3.6. *Let* $\mathbb{G} = \mathbb{R}^m \times \mathbb{R}^\ell$ *be the group in* (1.1)*. Then there exists a nontrivial abnormal length minimizing path if and only if the Métivier condition fails.*

Proof. Let $u \in L^2(0,1)$ be a nonzero abnormal length minimizing control. Since u must be normal-abnormal, it has the form (3.20) and we may assume the nondegeneration (3.21). Applying Proposition 3.4, we see that if $0 \neq w \in W$, then the dimension of $\operatorname{span}\{\langle Aw, \eta\rangle : \eta \in \mathbb{R}^m\}$ must be strictly less than ℓ. This means that the Métivier condition fails.

On the other side, if the Métivier condition fails, let $w \in \mathbb{R}^m \setminus \{0\}$ be such that $\eta \mapsto \langle Aw, \eta\rangle$ is not onto from $\mathbb{R}^m$ to $\mathbb{R}^\ell$. Then, by Proposition 3.4, we see that the curve $\gamma(s) = (sw, 0)$ is an abnormal minimizer. ☐

3.2. Failure of semiconcavity in two-step Carnot groups

3.2.1 Free groups

We show estimate (1.3) of Theorem 1.1. Namely, given $(x,t) = \gamma(1)$, final point of an abnormal minimizer γ, we want to show that there is $\sigma \in \wedge^2\mathbb{R}^m$ such that

$$\liminf_{\beta\to 0} \frac{d(x, t+\beta\sigma) - d(x,t)}{|\beta|} > 0 \tag{3.22}$$

Proof of (3.22). Let $(x,t) = \gamma(1) = (x(1), t(1))$, where γ is a normal-abnormal extremal. This means that γ originates from a control of the form $u(s) = \sum_{k=1}^p \cos(\lambda_k s)a_k + \sin(\lambda_k s)a_k^\perp + z$, where as usual we assume that $0 < \lambda_j < \lambda_k$ for all $j < k$ and moreover we have the singularity condition

$$\dim \operatorname{span}\{a_1, a_1^\perp, \ldots, a_p, a_p^\perp, z\} \leq m-2,$$

(here z may possibly vanish). Let $\operatorname{span}\{a_1, a_1^\perp, \ldots, a_p, a_p^\perp, z\} =: W$. Let $V := W^\perp = \operatorname{span}\{a_1, a_1^\perp, \ldots, a_p, a_p^\perp, z\}^\perp$. The singularity condition ensures that $\dim V \geq 2$. Let $\mathbb{F}_V := V \times \wedge^2 V$ be the subgroup generated by $V \times \{0\}$. We claim that for any nonzero bivector $\sigma \in \wedge^2 V$, we have

$$d\big(x, t+\beta\sigma\big) \geq d\big(x,t\big) + C|\beta|. \tag{3.23}$$

[1] Notice that $L_W := \{\langle Aw, \eta\rangle : w \in W,\ \eta \in \mathbb{R}^m\}$ in general is not a subspace of $\mathbb{R}^\ell$. This for instance happens if $\mathbb{R}^m \times \mathbb{R}^\ell = \mathbb{R}^4 \times \wedge^2\mathbb{R}^4$, $\langle x, Ay\rangle = x \wedge y$ and $W = \operatorname{span}\{e_1, e_2\}$. In such case, $e_1 \wedge e_3$ and $e_2 \wedge e_4 \in L_W$, but $e_1 \wedge e_3 + e_2 \wedge e_4 \notin L_W$.

To prove the claim, fix $\sigma \in \wedge^2 V \setminus \{0\}$ and let $\beta \in \mathbb{R}$. Take a minimizing control $u \in L^2(0,1)$ and let $\gamma = (x,t) : [0,1] \to \mathbb{F}_{m,2}$ be the corresponding minimizing path joining $(0,0)$ and $(x, t+\beta\sigma)$. Assume also the constant speed condition

$$|u(s)| = |\dot{x}(s)| = d\big(x, t+\beta\sigma\big) \quad \forall s \in [0,1].$$

Decompose orthogonally

$$\begin{aligned} u(s) &=: u_V(s) + u_W(s) \in V \oplus W \\ x(s) &= \int_0^s u =: x_V(s) + x_W(s) \in V \oplus W. \end{aligned} \tag{3.24}$$

Thus,

$$\begin{aligned} t(s) &= \frac{1}{2}\int_0^s (x_V + x_W) \wedge (u_V + u_W) \\ &= \frac{1}{2}\int_0^s x_V \wedge u_V + \frac{1}{2}\int_0^s x_V \wedge u_W + \frac{1}{2}\int_0^s x_W \wedge u_V + \frac{1}{2}\int_0^s x_W \wedge u_W \\ &=: t_V(s) + t_*(s) + t_W(s) \in \wedge^2 V \oplus (V \wedge W) \oplus \wedge^2 W, \end{aligned} \tag{3.25}$$

where we let

$$t_V(s) = \frac{1}{2}\int_0^s x_V \wedge u_V \in \wedge^2 V \quad \text{and} \quad t_W(s) = \frac{1}{2}\int_0^s x_W \wedge u_W \in \wedge^2 W.$$

By (3.10), the three terms in the last sum are pairwise orthogonal. The path $s \mapsto \gamma_V(s) = (x_V(s), t_V(s)) \in V \times \wedge^2 V$ is admissible in the Carnot group $V \times \wedge^2 V$ and the path γ_W is admissible in $W \times \wedge^2 W$.

Next we look at the final point of γ_V. Since

$$W \ni x = x(1) = x_V(1) + x_W(1) \in V \oplus W$$

we have $x_V(1) = 0$ and $x_W(1) = x$. Moreover, since

$$\wedge^2 W \oplus \wedge^2 V \ni t + \beta\sigma = t(1) = t_V(1) + t_*(1) + t_W(1) \in \wedge^2 V \oplus (V \wedge W) \oplus \wedge^2 W,$$

it must be $t_*(1) = 0 \in V \wedge W$, $t_V(1) = \beta\sigma \in \wedge^2 V$ and $t_W(1) = t \in \wedge^2 W$.

Ultimately, since the path γ_V connects the origin with $(0, \beta\sigma) \in \mathbb{F}_V = V \times \wedge^2 V$, we have

$$\int_0^1 |u_V| \geq d_{V \times \wedge^2 V}(0, \beta\sigma) \geq C|\beta|^{1/2}.$$

Moreover, since γ_W connects the origin with $(x,t) \in \mathbb{F}_W$, we have

$$\int_0^1 |u_W| \geq d_{\mathbb{F}_W}(x,t) \geq d_{\mathbb{F}_{V\oplus W}}(x,t) = d(x,t). \tag{3.26}$$

By the constant-speed assumption $|u(s)| = d(x, t+\beta\sigma)$ for all s,

$$\begin{aligned} d(x, t+\beta\sigma)^2 = \int_0^1 |u|^2 &= \int_0^1 |u_V|^2 + \int_0^1 |u_W|^2 \\ &\geq \Big(\int_0^1 |u_V|\Big)^2 + \Big(\int_0^1 |u_W|\Big)^2 \\ &\geq C|\beta| + d(x,t)^2. \end{aligned}$$

To conclude the proof, observe that if $(x,t) = (0,0)$, then $d(0,\beta\sigma) \geq C|\beta|^{1/2} \geq C|\beta|$ for $|\beta| < 1$. If instead $(x,t) \neq (0,0)$, then

$$d(x, t+\beta\sigma) \geq d(x,t)\Big(1 + \frac{C|\beta|}{d(x,t)^2}\Big)^{1/2} \geq d(x,t) + \frac{C'}{d(x,t)}|\beta|,$$

for small $|\beta|$. This proves (3.23). □

3.2.2 General two-step groups

Here we prove estimate (1.2), which shows that local semiconcavity fails for all two-step Carnot groups at abnormal points of the form $(w,0) \in \mathbb{R}^m \times \mathbb{R}^\ell$. The case of a general abnormal point seems to be technically more complicated and we do not discuss it. A procedure of lifting to a free group can be useful to discuss some specific examples, but the general case seems to require a deeper understanding of two-step Carnot group.

Proof of (1.2). Let $w \in \mathbb{R}^m$ be a unit vector such that the map $y \mapsto \langle Aw, y\rangle$ is not onto from $\mathbb{R}^m$ to $\mathbb{R}^\ell$. We claim that estimate (1.2) holds for any vector $\sigma \in \mathbb{R}^\ell \setminus \{0\}$ such that

$$\langle Aw, y\rangle \perp \sigma \quad \text{in } \mathbb{R}^\ell \quad \forall\, y \in \mathbb{R}^m. \tag{3.27}$$

Assume without loss of generality that $|\sigma| = 1$ in $\mathbb{R}^\ell$. Let $V := \operatorname{span}\{w\}^\perp =: W^\perp$ and

$$\mathbb{G}_V := \operatorname{span}\{(v, \langle Av', v''\rangle) : v, v', v'' \in V\}.$$

We claim that there is $C > 0$ such that $d(w, \beta\sigma) \geq 1 + C|\beta|$ uniformly in $\beta \in [-1,1]$. To show the claim, let $\gamma = (x,t) : [0,1] \to \mathbb{G}$ be a length minimizing constant-speed path, i.e. $|u(s)| = d(w,\beta\sigma)$ for all $s \in [0,1]$. Decompose

$$\begin{aligned} u(s) &= u_V(s) + u_W(s) \in V \oplus W \\ x(s) &= \int_0^s u =: x_V(s) + x_W(s) \in V \oplus W. \end{aligned} \tag{3.28}$$

Thus,

$$\begin{aligned} t(s) &= \frac{1}{2}\int_0^s \langle x, Au\rangle \\ &= \frac{1}{2}\int_0^s \langle x_V, Au_V\rangle + \frac{1}{2}\int_0^s (\langle x_V, Au_W\rangle + \langle x_W, Au_V\rangle) + \frac{1}{2}\int_0^s \langle x_W, Au_W\rangle \\ &=: t_V(s) + (t(s) - t_V(s)). \end{aligned} \tag{3.29}$$

where we put $t_V(s) := \frac{1}{2}\int_0^s \langle x_V, Au_V\rangle$. Note that the curve $\gamma_V(s) = (x_V(s), t_V(s))$ is an admissible curve in $\mathbb{G}_V$. The decomposition (3.28) proves that $x_V(1) = 0$. Formula (3.29) and the orthogonality condition (3.27) tell that $\beta\sigma \perp t(1) - t_V(1)$. Therefore, the required equality $t(1) = \beta\sigma$ implies that

$$|t_V(1)|^2 = \Big|\beta\sigma - (t(1) - t_V(1))\Big|^2 = \beta^2 + |t(1) - t_V(1)|^2 \geq \beta^2,$$

because $|\sigma| = 1$. Standard properties of two-step groups give

$$\text{length}_{\mathbb{G}_V}(\gamma_V) = \int_0^1 |u_V(s)|ds \geq C|\beta|^{1/2}.$$

A second obvious estimate concerns the curve $\zeta(s) := \langle x(s), w\rangle$. Since it satisfies $\zeta(0) = 0$ and $\zeta(1) = 1$, we have

$$\int_0^1 |u_W| = \int_0^1 |\dot\zeta| \geq 1.$$

To conclude the argument, starting from the constant speed property of $\gamma = (x,t)$ and using Cauchy-Schwarz, we find

$$\begin{aligned} d(w,\beta\sigma)^2 &= \int_0^1 |u|^2 = \int_0^1 |u_W|^2 + \int_0^1 |u_V|^2 \\ &\geq \Big(\int_0^1 |u_W|\Big)^2 + \Big(\int_0^1 |u_V|\Big)^2 \\ &\geq 1 + C|\beta| = d(w,0)^2 + C|\beta| \end{aligned}$$

and the proof is concluded. □

3.3. Horizontal semiconcavity estimates at abnormal points in free groups

By the results in [CR08,FR10], in a small neighborhood of the final point $\gamma(1) = (x,t)$ of a strictly normal minimizer, the distance from the origin is semiconcave. This estimate fails if γ is abnormal. However, a horizontal version of the semiconcavity property persists at abnormal points, at least in free groups. Indeed, if $\mathbb{F}_m$ is the free two-step group with m generators, for all $(x,t) = \gamma(1)$, where γ is abnormal length-minimizing on $[0,1]$, $\gamma(0) = (0,0)$ and $d(x,t) = 1$ there are positive constants C and δ so that

$$\sup_{y\in\mathbb{R}^m,\ |y|\leq\delta} \frac{d\big(e^{y\cdot X}(x,t)\big) + d\big(e^{-y\cdot X}(x,t)\big) - 2d(x,t)}{|y|^2} \leq C. \tag{3.30}$$

Here $y \cdot X := \sum_{j=1}^m y_j X_j$ and $e^{y\cdot X}(x,t)$ denotes the value at time $t = 1$ of the integral curve of $y \cdot X$ leaving from (x,t). We do not know whether or not such estimate holds uniformly in (x,t) on the unit sphere. We plan to come back to such problem in a further paper.

Estimate (3.30) can be proved by an induction argument and the discussion below is devoted to the proof of such statement.

Step 1. Let us start by observing that if $w_1, \dots, w_d$ is an orthonormal basis of a d-dimensional subspace $W \subset \mathbb{R}^m$ and $\mathbb{G}_W := W \times \wedge^2 W$ is a free subgroup of $\mathbb{F}_m := \mathbb{R}^m \times \wedge^2\mathbb{R}^m$, then for any point $(x,t) \in \mathbb{G}_W$ we have the estimate

$$d_{\mathbb{F}_m}(x,t) = d_{\mathbb{G}_W}(x,t) = d_{\mathbb{F}_d}(\xi,\tau) \tag{3.31}$$

where in the last equality we denoted $\xi_j = \langle x, w_j\rangle$ and $\tau_{jk} = \langle w_j \wedge w_k, t\rangle$ for $j = 1,\dots,d$. The $\leq$ in the first equality of (3.31) follows from the fact that $\mathbb{G}_W$ is a subgroup of $\mathbb{F}_m$. The $\geq$ holds because

(a) If $u \in L^2((0,1),\mathbb{R}^m)$ is a control in $\mathbb{F}_m$ such that the curve γ_u connects the origin with $(x,t) \in \mathbb{G}_W \subset \mathbb{F}_m$, then, the orthogonal projection $u_W \in L^2((0,1),W)$ is admissible in $\mathbb{F}_W$ and the corresponding curve γ_W connects the origin with $(x,t) \in \mathbb{G}_W$.

(b) $\operatorname{length}_{\mathbb{G}_W}(\gamma_W) \le \operatorname{length}_{\mathbb{F}_m}(\gamma)$.

Note that the $\ge$ inequality in (3.31) may fail if we change $\mathbb{F}_m$ with a nonfree two-step Carnot group $\mathbb{G}$. This can be seen by considering the group $\mathbb{R}^5 \times \mathbb{R} = \mathbb{G}$ with operation

$$(x_1, x_2.x_3.x_4, t) \cdot (\xi_1, \xi_2, \xi_3, \xi_4, t) = \Big(x+\xi, t+\tau+\frac{1}{2}(x_1\xi_2 - x_2\xi_1) + \frac{\alpha}{2}(x_3\xi_4 - x_4\xi_3)\Big)$$

with $\alpha > 1$ and its subgroup $\mathbb{G}_W := \mathbb{G}_{\operatorname{span}\{e_1,e_2\}} = \{(x_1,x_2,0,0,t)\}$. Here it turns out that $d_{\mathbb{G}}(0,0,0,0,\beta) = \sqrt{4\pi|\beta|/\alpha} < \sqrt{4\pi|\beta|} = d_{\mathbb{G}_W}(0,0,0,0,\beta)$.

Step 2. Let us look at estimate (3.30) for $m=3$. In such case abnormal points in the unit sphere are of the form $(x,t) = (w,0)$ for some $w \in \mathbb{R}^3$ with unit norm. Then, any vector $y \in \mathbb{R}^3$ can be written in the form $y = \xi w + \eta v$, where $\xi, \eta \in \mathbb{R}$ and $v \perp w$ is a suitable unit vector. Therefore, we have

$$\begin{aligned} e^{y\cdot X}(w,0) &= (w,0)\cdot \operatorname{Exp}(y\cdot X) = (w,0)\cdot(y,0) \\ &= (w,0)\cdot(\xi w + \eta v, 0) \in \mathbb{G}_{\operatorname{span}\{w,v\}} = \{(\xi w + \eta v, \tau w \wedge v) : (\xi,\eta,\tau) \in \mathbb{R}^3\} \end{aligned}$$

(here Exp denotes the standard Exponential map, see [BLU07, Definition 1.2.25]). Thus all points involved in the estimate belong to a subgroup which is isomorphic to the Heisenberg group $(\mathbb{H}_1, \circ)$. Therefore we have

$$\begin{aligned} & d\big((w,0)\cdot(\xi w+\eta v,0)\big) + d\big((w,0)\cdot(-\xi w - \eta v, 0)\big) - 2d(w,0) \\ &= d_{\mathbb{H}_1}((1,0,0)\circ(\xi,\eta,0)) + d_{\mathbb{H}_1}((1,0,0)\circ(-\xi,-\eta,0)) - 2d_{\mathbb{H}_1}(1,0,0) \le C(\xi^2+\eta^2), \end{aligned}$$

by the local semiconcavity of the distance in the Heisenberg group ([CR08, FR10]). Since this estimate is uniform as $v \in \mathbb{R}^3$ is a unit vector orthogonal to w, the statement in $\mathbb{F}_3$ follows easily.

Step 3. Next we describe the induction step. Assume that the estimate holds for $\mathbb{F}_{m-1}$ and let us look at $(x,t) = \gamma(1) \in \mathbb{F}_m$ with $d(x,t) = 1$, $\gamma(0) = (0,0)$ where γ is an abnormal length-minimizer. Let $W \subset \mathbb{R}^m$ be the associated subspace introduced in (3.16) and assume that $w_1, \dots, w_d$ is an orthonormal basis of W. The singularity condition means that $d \le m-2$. Moreover, any vector $y \in \mathbb{R}^m$ can be written in the form $y = \sum_{j=1}^d \xi_j w_j + \eta v$, where $v \perp W$ is a suitable unit vector depending on y (but we will get estimates which are uniform in $v \perp W$, $|v| = 1$). Therefore, we have

$$\begin{aligned} (x,t)\cdot \operatorname{Exp}(y\cdot X) &= (x,t)\cdot(y,0) \\ &= \Big(\sum_{j=1}^d x_j w_j, \sum_{j<k\le d} t_{jk} w_j \wedge w_k\Big)\cdot\Big(\sum_{j\le d}\xi_j w_j + \eta v, 0\Big) \in \mathbb{G}_{W\oplus \operatorname{span}\{v\}}. \end{aligned}$$

Thus, all involved points belong to a free subgroup which isomorphic to $\mathbb{F}_{d+1}$. If there is an abnormal length minimizer in such subgroup that connects the origin and (x,t), then, since $d+1 \le m-1$, using *Step 1* and arguing as in *Step 2*, we get the required statement (3.30). Otherwise, if any minimizer is normal, we can use [CR08] or [FR10] and we get again the desired estimate (3.30).

4. Lack of semiconcavity for the control distance in the Engel group

Let us consider the vector fields

$$X_1 = \partial_1 \quad \text{and} \quad X_2 = \partial_2 + x_1\partial_3 + \frac{x_1^2}{2}\partial_4.$$

It can be checked that X_1 and X_2 are left invariant on the Lie group in $\mathbb{E} = \mathbb{R}^4$ defined by the following law

$$x \cdot \xi = \Big(x_1 + \xi_1, x_2 + \xi_2, x_3 + \xi_3 + x_1\xi_2, x_4 + \xi_4 + \frac{x_1^2}{2}\xi_2 + x_1\xi_3\Big) \tag{4.1}$$

which is usally called *Engel group*. See [BLU07, p. 285]). Such vector fields belong to the model studied in the seminal paper [Sus96] on abnormal geodesics for rank two distributions and it is known that $\operatorname{Abn}\mathbb{E} = \{(0, x_2, 0, 0) : x_2 \in \mathbb{R}\}$. Therefore, by [CR08] and [FR10], we know that the distance from the origin d is locally semiconcave on $\mathbb{R}^4 \setminus \mathbb{R}e_2$. Here we show that d is not semiconcave at any point of the abnormal line. Moreover, we show that d is not semiconcave at such points even in the weaker horizontal sense.

In the papers [AS11, AS15] and [AT13] the explicit form of geodesics is established. In principle, our estimates could be obtained as a consequences of the mentioned results. However the form of such geodesics is rather involved and working with their explicit equations seems to be a rather difficult task.

Observe that the subset $\{(x_1, x_2, 0, x_4)\} \subset \mathbb{E}$ with the induced vector fields $X_1 = \partial_1$ and $X_2 = \partial_2 + \frac{x_1^2}{2}\partial_4$ can be identified with the Martinet subRiemannian system. See the discussion in the following Remarks 4.1 and 4.2.

Preliminarily we show that taken the constant control $\widetilde{u}(t) = (0, 1)$ for $t \in [0, 1]$, so that $E(\widetilde{u}) = (0, 1, 0, 0)$, we have

$$\operatorname{Im} dE(\widetilde{u}) = \operatorname{span}\{e_1, e_2, e_3\}. \tag{4.2}$$

We briefly check (4.2), by means of standard formula for the differential of the endpoint map. Following the notation in [ABB16], given $u \in L^2$, we denote by $P_s^t(x)$ the solution of $\frac{d}{dt}P_s^t(x) = \sum_j u_j(t)X_j(P_s^t(x))$, with $P_s^s(x) = x$. Thus, we have the well known formula

$$dE(u)v = \int \{v_1(t)dP_t^1(P_0^t(0))X_1(P_0^t(0)) + v_2(t)dP_t^1(P_0^t(0))X_2(P_0^t(0))\}dt.$$

See [Mon02, Rif14, ABB16]. At the point $u = \widetilde{u}$, we have $P_0^t x = e^{tX_2}x = \Big(x_1, x_2 + t, x_3 + tx_1, x_4 + \frac{x_1^2}{2}t\Big)$, so that

$$dP_t^1(x) = \begin{bmatrix} 1 & 0\;0\;0 \\ 0 & 1\;0\;0 \\ 1-t & 0\;1\;0 \\ (1-t)x_1 & 0\;0\;1 \end{bmatrix} \qquad P_0^t(0) = \begin{bmatrix} 0 \\ t \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad dP_t^1(P_0^t(0)) = \begin{bmatrix} 1 & 0\;0\;0 \\ 0 & 1\;0\;0 \\ 1-t & 0\;1\;0 \\ 0 & 0\;0\;1 \end{bmatrix}.$$

Therefore,

$$dE(u)v = \Big(\int_0^1 v_1(t)dt, \int_0^1 v_2(t)dt, \int_0^1 (1-t)v_1(t)dt, 0\Big) \tag{4.3}$$

which implies (4.2). The curve $x(t) = te_2$ is both normal and abnormal. It is abnormal because $dE(u)$ is not open. It is normal because the equality

$$\lambda_0\langle u, v\rangle_{L^2} + \big\langle(\lambda_1,\lambda_2,\lambda_3,\lambda_4), dE(u)v\big\rangle_{\mathbb{R}^4} = 0 \quad \forall\, v \in L^2 =: L^2((0,1),\mathbb{R}^2)$$

holds under the choice $\lambda_0 = -\lambda_2$ and $\lambda_1 = \lambda_3 = 0$.

It is very easy to show the failure of semiconcavity looking at the behavior of the distance in the orthogonal of $\operatorname{Im} dE(\widetilde{u})$, i.e. in $\operatorname{span}\{e_4\}$. This is shown by estimate (1.5), which we are now going to prove. A more precise version of the following proposition can be obtained as a consequence of [ABCK97] (see the remark after the proof).

Proof of (1.5). It suffices to show that there is $C_0 > 0$ such that

$$d(0,1,0,\lambda) - 1 \geq C_0|\lambda| \quad \text{for all } \lambda \text{ close to } 0. \tag{4.4}$$

To show estimate (4.4), let us consider the control problem $\dot\gamma = u_1(t)X_1(\gamma) + u_2(t)X_2(\gamma)$ with $\gamma(0) = (0,0,0,0)$ and $\gamma(1) = (0,1,0,\lambda)$, where $u = (u_1,u_2) \in L^2(0,1)$. Note that writing $\gamma = (\gamma_1,\gamma_2,\gamma_3,\gamma_4)$, we have

$$\gamma_3(1) = \int_{\gamma_0} x_2 dx_1 = 0 \quad \dot\gamma_4(1) = \frac{1}{2}\int_{\gamma_0} x_1^2 dx_2 = \lambda,$$

where we denoted $\gamma_0 := (\gamma_1,\gamma_2)$.

Let $(\gamma^\lambda)_{\lambda\in\mathbb{R}}$ be a family of curves $\gamma^\lambda : [0,1] \to \mathbb{R}^2$ satisfying $\gamma^\lambda(0) = (0,0)$, $\gamma^\lambda(1) = (0,1)$ and $\int_{\gamma^\lambda} x_1^2 dx_2 = 2\lambda$. Then

$$\begin{aligned}
|2\lambda| &= \Big|\int_{\gamma^\lambda} x_1^2 dx_2\Big| = \Big|\int_0^1 \gamma_1^\lambda(t)^2\dot\gamma_2^\lambda(t)dt\Big| \leq \sup_{t\in[0,1]} \gamma_1^\lambda(t)^2 \int_0^1 |\dot\gamma_2^\lambda(t)|dt \\
&\leq \sup_{(x_1,x_2)\in\gamma^\lambda([0,1])} x_1^2 \cdot \operatorname{length}(\gamma^\lambda) \\
&\leq 2\sup_{x\in\gamma^\lambda} x_1^2,
\end{aligned} \tag{4.5}$$

where we assumed without loss of generality that $\operatorname{length}(\gamma^\lambda) \leq 2$ for all $|\lambda|$ sufficiently small. Therefore, there is $t_\lambda \in (0,1)$ such that $\gamma_1^\lambda(t_\lambda) = |\lambda|^{1/2}$. Thus

$$\begin{aligned}
\operatorname{length}(\gamma_\lambda) &= \operatorname{length}(\gamma|_{[0,t_\lambda]}) + \operatorname{length}(\gamma|_{[t_\lambda,1]}) \\
&\geq |(|\lambda|^{1/2}, \gamma_2^\lambda(t_\lambda)) - (0,0)| + |(|\lambda|^{1/2}, \gamma_2^\lambda(t_\lambda)) - (0,1)| \\
&\geq \Big|\Big(|\lambda|^{1/2}, \frac{1}{2}\Big) - (0,0)\Big| + \Big|\Big(|\lambda|^{1/2}, \frac{1}{2}\Big) - (0,1)\Big| = 2\sqrt{\frac{1}{4} + |\lambda|},
\end{aligned}$$

and the claim follows. ☐

Remark 4.1. *If we let $x_3 = 0$ and we identify respectively (x_1,x_2,x_4) with $(y,x,z) \in \mathbb{R}^3$, an inspection of the proof above shows that we have proved the following estimate for the Martinet vector fields $X = \partial_x + \frac{y^2}{2}\partial_z$ and $Y = \partial_y$,*

$$\liminf_{z\to 0} \frac{d(1,0,z) - d(1,0,0)}{|z|} \gneqq 0.$$

If $z > 0$, then this estimate is contained in [ABCK97, eq. (4.31)], where it is shown that the intersection of the unit sphere with the abnormal set $y = 0$, has a parametrization of the form

$$x(t) = 1 - t + o(t) \quad \text{and} \quad z(t) = \frac{2}{3\pi^2}t + o(t) \quad \text{as } t \to 0+$$

If $z < 0$, then the absolute value in the first equality in the chain of estimates (4.5) *is very rough and our argument does not detect the logarithmic estimate proved by [ABCK97].*

4.1. Failure of horizontal semiconcavity at abnormal points

Here we prove the horizontal estimate (1.6).

Remark 4.2. *An inspection of the proof below shows that no information on the variable x_3 is used (we will not make any use of the first equation of* (4.6)*). Thus we get some more information on the distance for the Martinet vector fields $Y = \partial_y$ and $X = \partial_x + \frac{y^2}{2}\partial_z$. Namely we have the estimate*

$$\lim_{y\to 0} \frac{d(1,y,0) - d(1,0,0)}{y^2} = +\infty$$

Proof of (1.6). Since the case $x_2 = 0$ is trivial, without loss of generality it suffices to show the statement with $x_2 = 1$ and $y_2 = 0$. In such case we are able to prove that

$$\lim_{\lambda\to 0} \frac{d(e_2 \cdot \lambda e_1) - d(e_2)}{\lambda^2} = +\infty.$$

Note that $e_2 \cdot \lambda e_1 = e^{\lambda X_1}(0,1,0,0) = (\lambda,1,0,0)$. Any admissible curve in $\widetilde{\gamma} =: (\gamma, \gamma_3, \gamma_4) : [0,T] \mapsto \mathbb{E}$ is the lifting of its plane projection $\gamma : [0,T] \to \mathbb{R}^2$ with the constraints $\dot\gamma_3 = \gamma_1\dot\gamma_2$ and $\dot\gamma_4 = \frac{1}{2}\gamma_1^2\dot\gamma_2$. Thus, the requirements $\gamma_3(T) = \gamma_4(T) = 0$ can be written in the form

$$\int_\gamma x_1 dx_2 = 0 \quad \text{and} \quad \int_\gamma x_1^2 dx_2 = 0 \tag{4.6}$$

(the first equality will not be used in our argument).

Let us assume by contradiction that there exists a family of curves $x^\lambda : [0, T_\lambda] \to \mathbb{R}^2$ with $\lambda \in \mathbb{R}$ and a constant $C_0 > 0$ such that for all λ close to 0 all the following properties hold:

$$\begin{cases} x^\lambda(0) = (0,0), \qquad x^\lambda(T_\lambda) = (\lambda, 1) \\ |\dot x^\lambda| \le 1 \quad \text{a.e.} \\ T_\lambda - 1 \le C_0\lambda^2 \\ \displaystyle\int_0^{T_\lambda} x_1^\lambda(t)^2 \dot x_2^\lambda(t)dt = 0 \end{cases} \tag{4.7}$$

We will show that this produces the following contradiction. Letting

$$\text{(LHS)} := \int_{\{\dot x_2^\lambda > 0\}} (x_1^\lambda)^2 \dot x_2^\lambda = \Big| \int_{\{\dot x_2^\lambda \le 0\}} (x_1^\lambda)^2 \dot x_2^\lambda \Big| =: \text{(RHS)}.$$

we claim that there are $C_1, C_2 > 0$ such that if $|\lambda|$ is sufficiently small, then

$$\text{(RHS)} \le C_1\lambda^4 \quad \text{and} \quad \text{(LHS)} \ge C_2|\lambda|^3. \tag{4.8}$$

To get this contradiction, by symmetry it suffices to discuss the case $\lambda > 0$. The proof is articulated in several steps.

Step 1. First we prove the estimate $|\{t \in [0, T_\lambda] : \dot{x}_2^\lambda(t) \le 0\}| \le C_0\lambda^2$. This can be achieved easily because

$$1 = x_2^\lambda(T) - x_2^\lambda(0) = \int_0^{T_\lambda} \dot{x}_2^\lambda = \int_{\dot{x}_2^\lambda > 0} \dot{x}_2^\lambda + \int_{\dot{x}_2^\lambda < 0} \dot{x}_2^\lambda \le \int_{\dot{x}_2^\lambda > 0} \dot{x}_2^\lambda \le |\{\dot{x}_2^\lambda > 0\}|,$$

because $|\dot{x}_2^\lambda| \le |\dot{x}^\lambda| \le 1$. Thus $|\{\dot{x}_2^\lambda > 0\}| \ge 1$. Therefore, its complementary set satisfies

$$|\{\dot{x}_2^\lambda \le 0\}| = T_\lambda - |\{\dot{x}_2^\lambda > 0\}| \le 1 + C_0\lambda^2 - 1 = C_0\lambda^2.$$

Step 2. There is $C_1 > 0$ such that $\sup_{[0,T_\lambda]}(x_1^\lambda)^2 \le C_1\lambda^2$.

Let $\lambda > 0$ and define the positive number $q_\lambda = \max_{t\in[0,T_\lambda]} x_1^\lambda(t)/\lambda$. (An analogous discussion, left to the reader, can be given working with $p_\lambda := \min_{t\in[0,T_\lambda]} x_1^\lambda(t)/\lambda$). Assume that $q_\lambda \ge 2$, otherwise there is nothing to prove. Take a point $(\lambda q_\lambda, x_2^\lambda) \in \gamma^\lambda([0, T_\lambda])$. Then

$$\begin{aligned}
1 + C_0\lambda^2 &\ge \text{length}(x^\lambda) \ge d((\lambda q_\lambda, x_2^\lambda), (\lambda, 1)) + d((\lambda q_\lambda, x_2^\lambda), (0, 0)) \\
&\ge d((\lambda q_\lambda, x_2^\lambda), (\lambda, 1)) + d((\lambda q_\lambda, x_2^\lambda), (\lambda, 0)) \\
&\ge (\text{this quantity is minimal for } x_2^\lambda = \frac{1}{2}) \\
&\ge d((\lambda q_\lambda, 1/2), (\lambda, 1)) + d((\lambda q_\lambda, 1/2), (\lambda, 0)) \\
&= 2\sqrt{\frac{1}{4} + (q_\lambda - 1)^2\lambda^2}.
\end{aligned}$$

Comparing the first and the last term, we see that q_λ should be bounded uniformly for small positive λ.

Step 3. Estimate of (RHS):

$$\Big|\int_{\dot{x}_2^\lambda < 0} (x_1^\lambda)^2 \dot{x}_2^\lambda\Big| \le \sup_{[0,T_\lambda]} (x_1^\lambda)^2 |\{\dot{x}_2^\lambda < 0\}| \le C_1\lambda^2 \cdot C_0\lambda^2,$$

as desired.

The estimate of (LHS) is more delicate and we need some preliminary notation. Introduce the following rotation $\rho_\lambda : \mathbb{R}^2 \to \mathbb{R}^2$

$$\rho_\lambda \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \frac{1}{\sqrt{1+\lambda^2}} \begin{pmatrix} 1 & \lambda \\ -\lambda & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} (\xi_1 + \lambda\xi_2)/\sqrt{1+\lambda^2} \\ (\xi_2 - \lambda\xi_1)/\sqrt{1+\lambda^2} \end{pmatrix}. \tag{4.9}$$

Observe that $\rho_\lambda(0, \sqrt{1+\lambda^2}) = (\lambda, 1)$ for all λ. Define then

$$p_0 = 2\sqrt{2C_0} \tag{4.10}$$

and construct the following sets (we will work both with these sets and with their rotated through ρ_λ).

$$\widetilde{\ell}_\lambda := \{(\xi_1, \xi_2) : \xi_2 = \sqrt{1+\lambda^2}(1-\lambda)\}.$$

This is a horizontal line below the point $(0, \sqrt{1+\lambda^2})$ of an amount of order λ. Inside this line we fix the (rather short) segment

$$\begin{aligned}\widetilde{F}_\lambda &= \{(\xi_1, \xi_2) : \xi_2 = \sqrt{1+\lambda^2}(1-\lambda), |\xi_1| \le p_0\lambda^{3/2}\} \\ &= \Big\{ \big(\theta p_0 \lambda^{3/2}, \sqrt{1+\lambda^2}(1-\lambda)\big) : |\theta| \le 1 \Big\}\end{aligned}$$

and the tiny rectangle

$$\widetilde{R}_\lambda := \Big\{ \big(\theta_1 p_0 \lambda^{3/2}, \sqrt{1+\lambda^2}(1-\theta_2\lambda)\big) : |\theta_1| \le 1, \quad |\theta_2| \le 1 \Big\}.$$

which extends on top of $\widetilde{F}_\lambda$ of an amount approximately 2λ. Then, on the left and on the right of $\widetilde{R}_\lambda$ introduce the set

$$\widetilde{M}_\lambda := \Big\{ \big(\theta_1 p_0 \lambda^{3/2}, \sqrt{1+\lambda^2}(1-\theta_2\lambda)\big) : |\theta_1| \ge 1, \quad |\theta_2| \le 1 \Big\}.$$

Finally, on top of $\widetilde{R}_\lambda \bigcup \widetilde{M}_\lambda$ we have the half-plane

$$\widetilde{G}_\lambda := \Big\{ \big(\theta_1 p_0 \lambda^{3/2}, \sqrt{1+\lambda^2}(1-\theta_2\lambda)\big) : \theta_1 \in \mathbb{R}, \quad \theta_2 \le -1 \Big\}.$$

Correspondingly we have the rotated sets $\ell_\lambda := \rho_\lambda \widetilde{\ell}_\lambda$, $F_\lambda := \rho_\lambda \widetilde{F}_\lambda$, $R_\lambda := \rho_\lambda \widetilde{R}_\lambda$, and $M_\lambda = \rho_\lambda \widetilde{M}_\lambda$. The tiny rectangle R_λ is centered at the final point $(\lambda, 1)$.

Step 4. Under the choice of p_0 made in (4.10), we have for sufficiently small positive λ

$$x^\lambda([0, T_\lambda]) \cap M_\lambda = \varnothing \qquad \text{and} \quad x^\lambda([0, T_\lambda]) \cap G_\lambda = \varnothing.$$

We start with the proof of the first claim, which gives the more striking information, due to the power $\lambda^{3/2}$ in the horizontal size of R_λ. We work with the rotated curve $\xi^\lambda(t) = \rho_\lambda^{-1} x^\lambda(t)$. Such curve has length at most $1 + C_0\lambda^2$ and connects $(0,0)$ with $(0, \sqrt{1+\lambda^2})$. Assume by contradiction that there is a point belonging to $\widetilde{M}_\lambda \cap \xi^\lambda([0, T_\lambda])$. Such point has the form $\big(\theta_1 p_0 \lambda^{3/2}, \sqrt{1+\lambda^2}(1-\theta_2\lambda)\big)$, for some θ_1, θ_2 satisfying $|\theta_1| \ge 1$, and $|\theta_2| \le 1$. Therefore, the estimate on the length furnishes

$$\begin{aligned} 1 + C_0\lambda^2 &\ge \operatorname{length}(x^\lambda) \\ &\ge d\Big((0,0), (\theta_1 p_0 \lambda^{3/2}, \sqrt{1+\lambda^2}(1-\theta_2\lambda))\Big) \\ &\qquad + d\Big((\theta_1 p_0 \lambda^{3/2}, \sqrt{1+\lambda^2}(1-\theta_2\lambda)), (0, \sqrt{1+\lambda^2})\Big) \\ &\ge (\text{we minimize choosing } \theta_2 = 1) \\ &\ge \sqrt{\theta_1^2 p_0^2 \lambda^3 + (1+\lambda^2)(1-\lambda)^2} + \sqrt{\theta_1^2 p_0^2 \lambda^3 + (1+\lambda^2)\lambda^2} \\ &\ge 1 - \lambda + \sqrt{\theta_1^2 p_0^2 \lambda^3 + (1+\lambda^2)\lambda^2} \ge 1 - \lambda + \lambda\sqrt{1 + p_0^2\lambda}. \end{aligned}$$

Comparing the first and the last term, we see that this chain of inequality conflicts with the choice of p_0 made in (4.10), for small λ.

Next we show the second statement of *Step 4.* Let $\lambda > 0$ be a small number and assume by contradiction that there exists $\bar{x}^\lambda \in G_\lambda \cap x^\lambda([0, T_\lambda])$. The rotated point $\overline{\xi}^\lambda :=$

$\rho_\lambda^{-1}\overline{x}^\lambda$ has the form $(\theta_1 p_0\lambda^{3/2}, \sqrt{1+\lambda^2}(1-\theta_2\lambda))$ with $\theta_1 \in \mathbb{R}$ and $\theta_2 \le -1$. Thus, it must be $\bar{\xi}_2^\lambda \ge (1+\lambda)\sqrt{1+\lambda^2}$. Therefore

$$\begin{aligned}
1+C_0\lambda^2 &\ge d((0,0),(\bar{\xi}_1^\lambda,\bar{\xi}_2^\lambda)) + d((\bar{\xi}_1^\lambda,\bar{\xi}_2^\lambda),(0,\sqrt{1+\lambda^2}))\\
&\ge |\bar{\xi}_2^\lambda| + |\bar{\xi}_2^\lambda - \sqrt{1+\lambda^2}|\\
&\ge (1+\lambda)\sqrt{1+\lambda^2} + \big((1+\lambda)\sqrt{1+\lambda^2} - \sqrt{1+\lambda^2}\big)\\
&= (1+2\lambda)\sqrt{1+\lambda^2}.
\end{aligned}$$

Again, comparing the first and last term, we find a contradiction and *Step 4.* is accomplished.

Step 5. We claim that if $(x_1,x_2) \in F_\lambda$, then $x_2^\lambda \le 1-\frac{\lambda}{2}$. Here we use the fact that the segment F_λ is very short with respect to λ.

To check the claim, recall that $x \in F_\lambda$ means that there is $\theta \in [-1,1]$ such that $(x_1,x_2) = \rho_\lambda(\theta p_0\lambda^{3/2}, \sqrt{1+\lambda^2}(1-\lambda))$ Thus, using (4.9), we find

$$\bar{x}_2 = \frac{1}{\sqrt{1+\lambda^2}}\Big(-\theta p_0\lambda^{5/2} + \sqrt{1+\lambda^2}(1-\lambda)\Big)$$

and Step 5 is accomplished, if $\lambda > 0$ is sufficiently small.

Step 6. If $x=(x_1,x_2) \in R_\lambda$, then we have $x_1 \ge \frac{\lambda}{2}$.

This can be seen again by means of (4.9), which gives for suitable $\theta_1,\theta_2 \in [-1,1]$

$$x_1 = \frac{1}{\sqrt{1+\lambda^2}}\Big(\theta_1 p_0\lambda^{3/2} + \lambda\sqrt{1+\lambda^2}(1-\theta_2\lambda)\Big) \ge \frac{\lambda}{2},$$

for all positive λ sufficiently small.

Step 7. Lower estimate of (LHS).

Take λ and the corresponding curve x^λ. Let $t_\lambda \in [0,T_\lambda]$ be the unique time such that

$$x^\lambda(t_\lambda) \in F_\lambda \qquad x^\lambda(t) \notin F_\lambda \quad \forall t \in\,]t_\lambda, T_\lambda].$$

Note that $x^\lambda(t) \in R_\lambda$ for all $t \in [t_\lambda, T_\lambda]$. This follws fron the fact that the curve x^λ can intersect the line ℓ_λ only in the segment F_λ. Thus, after the time t_λ it should lie on top of such line. On the other side, by *Step 4.*, the curve cannot touch the "prohibited set" $M_\lambda \cup G_\lambda$. Therefore $x^\lambda([t_\lambda,T_\lambda]) \subset R_\lambda$. Therefore

$$\begin{aligned}
\int_{\dot{x}_2^\lambda>0}(x_1^\lambda)^2\dot{x}_2^\lambda &\ge \int_{[t_\lambda,T_\lambda]\cap\{\dot{x}_2^\lambda>0\}}(x_1^\lambda)^2\dot{x}_2^\lambda \ge \inf_{(x_1,x_2)\in R_\lambda} x_1^2\int_{[t_\lambda,T_\lambda]\cap\{\dot{x}_2^\lambda>0\}}\dot{x}_2^\lambda \ge \inf_{(x_1,x_2)\in R_\lambda} x_1^2\int_{[t_\lambda,T_\lambda]}\dot{x}_2^\lambda\\
&\ge \text{(By Step 6)} \ge \frac{\lambda^2}{4}(x_2^\lambda(T_\lambda) - x_2^\lambda(t_\lambda))\\
&\ge \text{(By Step 5)} \ \ge \frac{\lambda^2}{4}\Big(1-\Big(1-\frac{\lambda}{2}\Big)\Big) = \frac{\lambda^3}{8},
\end{aligned}$$

and the proof is concluded. $\square$

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